1.

METHODS 3 '09 (ii)

(a) $f(x,y) = 3xy^2 - 2x^2 - 3y^2 - 8x + 2$

$\frac{\partial f}{\partial x} = 3y^2 - 4x - 8 \quad = 0$ when $3y^2 = 4x + 8$, $y^2 = \frac{4}{3}x + \frac{8}{3}$

$\frac{\partial f}{\partial y} = 6xy - 6y \quad = 0$ when $y = 0 \rightarrow (-2,0)$ or $x = 1 \rightarrow (1, \pm 2)$

$3y^2 = 12$, $y^2 = 4$

i.e. points $(-2,0), (1,2), (1,-2)$

$f_{xx} = -4$
$f_{yy} = 6x - 6$
$f_{xy} = 6y$

	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}^2$	
$(-2,0)$:	-4	-18	0	> 0	max
$(1,2)$:	-4	0	12	$<$	sadd
$(1,-2)$:	-4	0	-12	$<$	sadd.

(b) $f(x,y,z,\lambda,\mu) = x^2 + y^2 + z^2 - \lambda(x^2 + y^2 - z^2) - \mu(x + y - z)$

- $f_x = 2x - 2\lambda x - \mu = 0 \Rightarrow 2x(1-\lambda) = \mu$
- $f_y = 2y - 2\lambda y - \mu = 0 \Rightarrow 2y(1-\lambda) = \mu$ $\Big\}$ $x = y$
- $f_z = 2z + 2z\lambda + \mu = 0 \Rightarrow 2z(1+\lambda) = -\mu$
- $z^2 = x^2 + y^2 \Rightarrow z^2 = 2x^2$
- $z = x + y + 2 \Rightarrow z = 2x + 2 \quad \Rightarrow z^2 = 4x^2 + 8x + 4 = 2x^2$

$\Rightarrow 2x^2 + 8x + 4 = 0$

i.e. $x^2 + 4x + 2 = 0$

i.e. ~~$x = -2 \pm \sqrt{4-2}$~~

$x = \frac{-4 \pm \sqrt{16 - 8}}{2}$

$= -2 \pm \sqrt{2}$

$\Rightarrow z = -2 \pm 2\sqrt{2}$

$z = -2 \pm 2\sqrt{2}$

$\Rightarrow \cancel{2}(-2 \pm \sqrt{2})(1-\lambda) = -\cancel{2}(-2 \pm 2\sqrt{2})(1+\lambda)$

$1 - \lambda = -\frac{(-2 \pm 2\sqrt{2})}{(-2 \pm \sqrt{2})}(1+\lambda)$

2(a) $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] = 0$$

if $\frac{\partial F}{\partial x} = 0$, check $\frac{d}{dx}\left[F - y'\frac{\partial F}{\partial y'}\right]$

$$= \overbrace{\frac{\partial F}{\partial x}}^{0} + \frac{\partial F}{\partial y}y' + \cancel{\frac{\partial F}{\partial y'}\frac{dy'}{dx}} - \cancel{y''\frac{\partial F}{\partial y'}} - y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

$$= y' \cdot 0 = 0.$$

(b) $A[y] = \frac{1}{2}\int_0^T (y')^2 - \omega^2 y^2 \, dx \qquad y(0) = a \quad y(T) = 0$

$$F = \frac{1}{2}\left[(y')^2 - \omega^2 y^2\right]$$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] = -\omega^2 y - \frac{d}{dx}\left[y'\right] = 0$$

i.e. $y'' + \omega^2 y = 0. \qquad y(0) = a \quad y(T) = 0.$

$\Rightarrow y = A\cos\omega x + B\sin\omega x \qquad y(0) = a \quad y(T) = 0$

$\Rightarrow a = A$

& $0 = a\cos\omega T + B\sin\omega T \quad \Rightarrow \quad B = -\frac{a\cos\omega T}{\sin\omega T}$

$\Rightarrow y = a\cos\omega x + (a\cot\omega T)\sin\omega x$

$$A[y_c] = \frac{1}{2}\int_0^T \omega^2\left[-a\sin\omega x + a\cot\omega T\cos\omega x\right]^2 - \omega^2\left[a\cos\omega t + a\cot\omega T\sin\omega x\right]^2$$

$$= \frac{1}{2}\int_0^T\left[\omega^2 a^2\right.$$

$$A[y] = \frac{1}{2}\int_0^T (y')^2 - \omega^2 y^2 \, dx$$

$$= \frac{1}{2}\int_0^T (y')^2 \, dx - \frac{1}{2}\int_0^T \omega^2 y^2 \, dx$$

$$= \left[\frac{1}{2}(y'y)\right]_0^T - \underbrace{\frac{1}{2}\int_0^T y''y - \omega^2 y^2 \, dx}_{0}$$

$$= \left[\frac{1}{2}(y'y)\right]_0^T$$

$$= \frac{1}{2}\left[y'(T)\underset{0}{y(\cancel{T})} - y'(0)\,y(0)\right]$$

$$= -\frac{1}{2}\left[y'(0)\,a\right]$$

$$= \frac{1}{2}a^2\omega\cot\omega T$$

3 (a) $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = z^2$ $\qquad z = f(x) \quad y = 0$

$\downarrow$

$x = s$
$y = 0$
$z = f(s)$

$\dfrac{dx}{dt} = 1 \qquad \dfrac{dy}{dt} = 1 \qquad \dfrac{dz}{dt} = z^2$

$x = t + x_0 \qquad y = t + y_0 \qquad$ ~~$z = \ldots$~~

$\Downarrow$

~~$x = \ldots$~~

~~$x_0 = \ldots$~~

$\underline{x = s + t} \qquad \underline{y = t}$

$\int \frac{1}{z^2}\, dz = \int dt$

$\Rightarrow t = -\frac{1}{z} + z_0$

$\Rightarrow t = -\frac{1}{z} + \frac{1}{f(s)}$

$\Rightarrow \frac{1}{z} = \frac{1}{f(s)} - t = \dfrac{1 - t f(s)}{f(s)}$

$\Rightarrow z = \dfrac{f(s)}{1 - t f(s)}$

$\Rightarrow z = \dfrac{f(x-y)}{1 - y f(x-y)}$

(b) ~~$\dfrac{dx}{dt} = y \quad \dfrac{dy}{dt} = -2xy \quad \dfrac{dz}{dt} = 2xz$~~

$\dfrac{dx}{y} = \dfrac{dy}{-2xy} = \dfrac{dz}{}$

$\dfrac{dy}{dx} = -2x \Rightarrow y = -x^2 + c \Rightarrow \underline{y + x^2 = k}$

So let $\psi = x$
$\varphi = y + x^2$

~~$\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial \psi} \quad \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial \varphi}\dfrac{\partial \varphi}{\partial y} = \dfrac{\partial z}{\partial \varphi}$~~

~~$\Rightarrow (\varphi - \psi^2)\left(\dfrac{\partial z}{\partial \psi}\right) - 2\psi(\varphi - \psi^2)\dfrac{\partial z}{\partial \varphi} = 2\psi z$~~

Let $\psi = x, \quad \varphi = y + x^2$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \psi}\frac{\partial \psi}{\partial x} + \frac{\partial z}{\partial \varphi}\frac{\partial \varphi}{\partial x} = \frac{\partial z}{\partial \psi} + 2\psi \frac{\partial z}{\partial \varphi}$$

$$\frac{\partial z}{\partial y} = \cancel{\frac{\partial z}{\partial \psi}\frac{\partial \psi}{\partial y}}^{0} + \frac{\partial z}{\partial \varphi}\frac{\partial \varphi}{\partial y} = \frac{\partial z}{\partial \varphi}$$

$$\Rightarrow (\varphi - \psi^2)\left[\frac{\partial z}{\partial \psi} + 2\psi \frac{\partial z}{\partial \varphi}\right] - 2\psi(\varphi - \psi^2)\frac{\partial z}{\partial \varphi} = 2\psi z$$

$$\Rightarrow (\varphi - \psi^2)\frac{\partial z}{\partial \psi} = 2\psi z$$

$$\frac{\partial z}{\partial \psi} = \frac{2\psi z}{\varphi - \psi^2}$$

$$\ln z = -\ln(\varphi - \psi^2) + f(\varphi)$$

$$\ln z = -\ln y + f(y + x^2)$$

$$z = y\,\tilde{f}(y + x^2)$$

(c) $$\frac{dx}{xz} = \frac{dy}{-yz} = \frac{dz}{x^2 - y^2}$$

$$\frac{dx}{dt} = xz \qquad \frac{dy}{dt} = -yz \qquad \frac{dz}{dt} = x^2 - y^2$$

$$y\frac{dx}{dt} + x\frac{dy}{dt} = 0 \Rightarrow xy = c$$

$$x^2 + y^2 - z^2 = 0$$
$$2x\dot{x} + 2y\dot{y} - 2z\dot{z} = 0$$

$$2x^2 z - 2y^2 z - 2zx$$

$$x^2 + y^2 - z^2 = c$$

$$\Rightarrow xy = f(x^2 + y^2 - z^2)$$ by Lagrange.

4. D'Alembert's Sol^n.

$$z_{xx} = \frac{1}{c^2} z_{tt}$$

Look for sol^{n}s of form $f(x+mt) \Rightarrow m = \pm c$

$$\Rightarrow z = f(x+ct) + g(x-ct)$$

when $t=0$, $f(x) + g(x) = F(x)$

$$cf'(x) - cg'(x) = G(x)$$

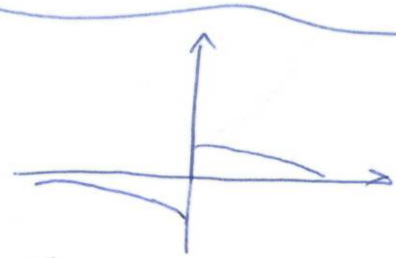
$$\Rightarrow \int_\alpha^x G(s)\,ds = cf(x) - cg(x)$$

$$\Rightarrow f(x) = \frac{1}{2}\left(F(x) + \frac{1}{c}\int_\alpha^x G(s)\,ds\right)$$

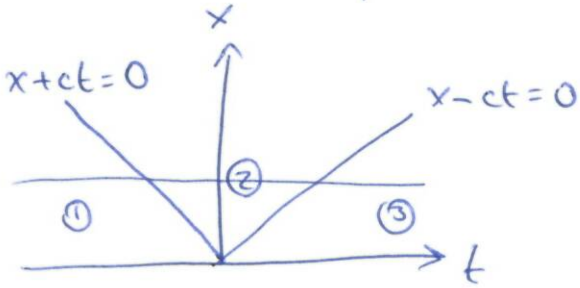
$$g(x) = \frac{1}{2}\left(F(x) - \frac{1}{c}\int_\alpha^x G(s)\,ds\right)$$

$$\Rightarrow f(x+ct) + g(x-ct) = \frac{1}{2}\left[F(x+ct) + F(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds$$

$F(x) = 0 \qquad G(x) = \begin{cases} -e^x & x<0 \\ e^{-x} & x>0 \end{cases}$



$$\Rightarrow z(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds$$



① $z(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} -e^s\,ds$

eh.

5. $\theta_{xx} = \frac{1}{\alpha^2}\theta_t$ $\quad\quad$ $\theta_x(-L,t) = \theta_x(L,t) = 0$.

$\theta = X(x)T(t)$

$\to X''T = \frac{1}{\alpha^2}XT' \Rightarrow \frac{X''}{X} = \frac{T'}{\alpha^2 T} = \lambda = -p^2$

$X = A\cos px + B\sin px$ $\quad\quad$ $T' + p^2\alpha^2 T = 0$

$T = e^{-p^2\alpha^2 t}$

$-Ap\sin pL + Bp\cos pL = 0$

$-Ap\sin pL + Bp\cos pL = 0$

$\Rightarrow 2Ap\sin pL = 0 \Rightarrow pL = n\pi$

$\Rightarrow 2Bp\cos pL = 0 \Rightarrow B = 0$.

$\to X = A_n\cos\frac{n\pi x}{L}$

$T = e^{-n^2\pi^2\alpha^2 t/L^2}$

$\Rightarrow \theta(x,t) = A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right)\exp(-n^2\pi^2\alpha^2 t/L^2)$

At $t=0$, $\theta(x,0) = A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right)$

$2a = \int_{-L}^{L}\theta(x,0)\,dx = A_0 2L \Rightarrow A_0 = \frac{a}{L}$

$\int_{-L}^{L}\theta(x,0)\cos\frac{m\pi x}{L}dx = \int_{-L}^{L}\frac{a}{L}\cos\frac{m\pi x}{L}dx \to 0 + LA_m$

$= 2\int_0^a\cos\frac{m\pi x}{L}dx = \frac{2L}{m\pi}\left[\sin\frac{m\pi a}{L}\right] \to A_m = \frac{2}{m\pi}\sin\frac{m\pi a}{L}$

$\to \theta(x,t) = \frac{a}{L} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\frac{n\pi a}{L}\cos\frac{n\pi x}{L}\exp(-n^2\pi^2\alpha^2 t/L^2)$

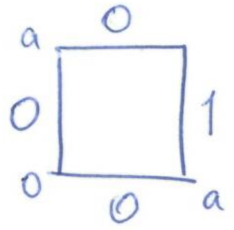
as $a\to 0$ $\frac{L\theta(x,t)}{a} = 1 + \sum_{n=1}^{\infty}\frac{\cancel{L}}{\cancel{a}}2\cos\frac{n\pi x}{L}\exp()$. $\quad\square$.

6. $u(x,y) = X(x)Y(y) = 0$

$X''Y + Y''X = 0$

$\frac{X''}{X} = \frac{-Y''}{Y} = \lambda$.

$\lambda > 0$, exponential
$\lambda < 0$, oscillatory X
$\lambda = 0$ linear X.



Oscillatory soln in Y $Y'' + Y = \lambda$ $\lambda > 0 = p^2$

$X'' - p^2 X = 0$

$\Rightarrow X = A\cosh px + B\sinh px$

At $x=0$, $X=0 \Rightarrow A = 0 \Rightarrow X = B\sinh px$

At $x=a$, $X=0 \Rightarrow pa = n\pi \Rightarrow p = \frac{n\pi}{a}$.

for Y, $Y = C\cos px + D\sin px$

At $y=0$, $Y=0 \Rightarrow$ $\cancel{Y = D\sin n\pi}$ $C=0$

$$\Rightarrow u = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

Obtain A_n by $u(x,a) = 1 \Rightarrow 1 = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a}\sinh(n\pi)$

$$\Rightarrow \int_0^a \left(\sin\frac{m\pi x}{a}\right)dx = \frac{1}{2} a A_n \sinh(n\pi)$$

i.e. $\left[\frac{a}{m\pi}\cos\frac{m\pi x}{a}\right]_a^0 = \frac{a}{m\pi} - \frac{a}{m\pi}\cos m\pi = \frac{a}{m\pi}\left(1-(-1)^m\right)$

$= 0$ for even m
$= \frac{2a}{m\pi}$ for odd.

$\Rightarrow$ let $\lambda_m = 2m+1 \Rightarrow A_n = \frac{4}{\lambda m\pi \sinh \lambda\pi}$

1(a) $f(x,y) = 3xy^2 - 2x^2 - 3y^2 - 8x + 2$

$f_x = 3y^2 - 4x - 8 = 0 \Rightarrow y^2 = \frac{8+4x}{3}$

$f_y = 6xy - 6y = 0 \Rightarrow y = 0$ or $x = 1$.

$\left.\begin{array}{l} \text{If } y=0,\ x=2 \\ \text{If } x=1,\ y=\pm 2 \end{array}\right\}$ 3 pts: $(-2,0)$; $(1,2)$; $(1,-2)$

Need to examine $\Delta = f_{xx}f_{yy} - f_{xy}^2$.

$\left.\begin{array}{l} f_{xx} = -4 \\ f_{yy} = 6x-6 \\ f_{xy} = 6y \end{array}\right\}$ $\Delta = -4(6x-6) - 36y^2$

At $(-2,0)$: $\Delta = -4(\overset{-18}{\cancel{6}}) = +24$ max

$(1,2)$: $\Delta = -36\cdot 4$ saddle

$(1,-2)$: $\Delta = -36\cdot 4$ saddle

(b) $f = x^2+y^2+z^2 - \lambda(x^2+y^2-z^2) - \mu(x+y-z)$

Solve $\partial f/\partial x = \partial f/\partial y = \partial f/\partial z = 0$:

- $2x - 2\lambda x - \mu = 0$ $\qquad 2x(1-\lambda) = \mu$
- $2y - 2\lambda y - \mu = 0$ $\qquad 2y(1-\lambda) = \mu$
- $2z + 2\lambda z + \mu = 0$ $\qquad 2z(1+\lambda) = -\mu$
- $z^2 = x^2 + y^2$
- $z = x + y + 2$ $\qquad x = y \Rightarrow z = 2x+2$

$2x(1-\lambda) = -2(2x+2)(1+\lambda)$

$(1-\lambda) = (1+\lambda)\cdot \underbrace{-\frac{2x+2}{x}}$

$= -\frac{-2\pm 2\sqrt{2}}{-2\pm\sqrt{2}}$ check.

$z^2 = 4x^2 + 8x + 4 = 2x^2$

$\rightarrow 2x^2 + 8x + 4 = 0$

$x^2 + 4x + 2 = 0$

$x = \frac{-4\pm\sqrt{16-8}}{2} = -2\pm\sqrt{2}$

2 (a) EL eq^n: $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$.

If $\partial F/\partial x = 0$, consider.

$$\frac{d}{dx}\left[F - y'\frac{\partial F}{\partial y'}\right].$$

$$\cancel{= \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial y'}\frac{dy'}{dx}}$$

$$\cancel{= \frac{\partial F}{\partial y}y' - y''\frac{\partial F}{\partial y'} + \frac{\partial F}{\partial y'}y'' - y'\frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right]}$$

$$= \cancel{\frac{\partial F}{\partial x}}^{\,0} + \frac{\partial F}{\partial y}y' + \cancel{\frac{\partial F}{\partial y'}y''} - \cancel{y''\frac{\partial F}{\partial y'}} + \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right]y'$$

$$= \frac{\partial F}{\partial y}y' + \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right]y'$$

$$= y'\left[\frac{\partial F}{\partial y} + \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] = 0 \qquad (\text{E.L eq}^n!)$$

(b) $F = \frac{1}{2}\left[(y')^2 - \omega^2 y^2\right]$

$F - y'\frac{\partial F}{\partial y'} = \text{const}$: $\quad \frac{1}{2}\left[(y')^2 - \omega^2 y^2\right] - \frac{1}{2}y'\left[2y'\right] = c$

$$-\frac{1}{2}(y')^2 - \omega^2 y^2 = c$$

$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \Rightarrow -y\omega^2 - \frac{d}{dx}(y') = 0 \Rightarrow \omega^2 y + y'' = 0$

$$\Rightarrow A\cos\omega t + B\sin\omega t = \cancel{0}\, y$$

$y(0) = a \Rightarrow A = a$. $\quad y(T)$: $0 = a\cos\omega T + B\sin\omega t$

$$\Rightarrow B = -\frac{a\cos\omega T}{\sin\omega T}$$

2b contd.

$$A[y] = \frac{1}{2}\int_0^T (y')^2 - \omega^2 y^2 \, dx$$

$$= \cancel{\frac{1}{2}\int_0^T \left[[y' + \omega y][y' - \omega y] \right] dx}$$

$$= \cancel{\frac{1}{2}\left[[y' + \omega y] \right.}$$

$$= \frac{1}{2}\int_0^T (y')(y')\,dx - \frac{1}{2}\int_0^T \omega^2 y^2 \, dx$$

$$= \frac{1}{2}[y'y]_0^T - \frac{1}{2}\int_0^T y''y \, dx - \frac{1}{2}\omega^2 y^2 \, dx$$

$$= \frac{1}{2}[y'y]_0^T - \frac{1}{2}\int_0^T \underbrace{y''y + \omega^2 y^2 \, dx}_{=0}$$

$$= \frac{1}{2}[y'y]_0^T \qquad \text{for extremal curve.}$$

$$= \frac{1}{2}y'(T)\underbrace{y(T)}_{0} - \frac{1}{2}y(0)y'(0)$$

$$= -\frac{1}{2}a y'(0)$$

$$= -\frac{1}{2}a\left[\overset{a}{\cancel{A}}\cos\omega t + B\sin\omega t\right]'(0)$$

$$= -\frac{1}{2}a\left[\cancel{\omega} B \cancel{\omega\cos\omega}\right]$$

$$= \underline{\frac{1}{2}a^2\omega\cot\omega T}.$$

3 (a) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z^2$ $\qquad z = f(x) \quad y = 0$

$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = 1 \qquad \frac{dz}{dt} = z^2$

$\int \frac{1}{z^2}\,dz = \int dt$

$x = t + x_0 \qquad y = t + y_0 \qquad -\frac{1}{z} = t + z_0$

$-\frac{1}{z} = t - \frac{1}{f(s)}$

$\underline{x = t + s} \qquad \underline{y = t}$

$x = s$

$y = 0$

$z = f(s)$

$-\frac{1}{z} = t - \frac{1}{f(s)}$

$-\frac{1}{z} = \frac{f(s)t - 1}{f(s)}$

$z = \frac{-f(s)}{f(s)t - 1} = \underline{\frac{f(s)}{1 - f(s)t}}$

$z = \underline{\frac{f(x-y)}{1 - f(x-y)y}}$.

(b) $y\frac{\partial z}{\partial x} - 2xy\frac{\partial z}{\partial y} = 2xz$

$\frac{dx}{dt} = y \qquad \frac{dy}{dt} = -2xy \qquad \frac{dz}{dt} = 2xz$

or $\frac{dx}{y} = \frac{dy}{-2xy} = \frac{dz}{2xz}$

$\Rightarrow -(2xy)\,dx = y\,dy$

$-2x\,dx = dy$

$-x^2 = y + c. \quad \Rightarrow \quad y + x^2 = k.$

Let $k = y + x^2$. Change vars from x and y to x and k

$$\Rightarrow y\frac{\partial z}{\partial x} - 2xy\frac{\partial z}{\partial y} = 2xz$$

~~becomes~~ $\cancel{(k-x^2)\frac{\partial z}{\partial x} + 2x(k-x^2)\frac{\partial z}{\partial k}\frac{\partial k}{\partial y} = 2xz}$

$\cancel{y\left(\frac{\partial z}{\partial x} + 2x\frac{\partial z}{\partial k}\right) - 2xy\frac{\partial z}{\partial k} = 2xz}$

$$\Rightarrow \frac{\partial z}{\partial x} \Rightarrow \frac{\partial z}{\partial k}\frac{\partial k}{\partial x} + \frac{\partial z}{\partial x} \qquad \frac{\partial k}{\partial x} = 2x$$

$$\frac{\partial z}{\partial y} \Rightarrow \frac{\partial z}{\partial k}\frac{\partial k}{\partial y} + \cancel{\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}} \to 0 \qquad \frac{\partial k}{\partial y} = 1$$

Becomes $y\left[2x\frac{\partial z}{\partial k} + \frac{\partial z}{\partial x}\right] - 2xy\left[\frac{\partial z}{\partial k}\right] = 2xz$

$$\rightarrow y\frac{\partial z}{\partial x} = 2xz$$

$$(k - x^2)\frac{\partial z}{\partial x} = 2xz$$

$$\frac{\partial z}{\partial x} = \frac{2xz}{k - x^2}$$

$$\rightarrow \cancel{z = 2\ln} \int \frac{1}{z}\,dz = \int \frac{2x}{k - x^2}\,dx$$

$$\ln z = -\ln(k - x^2) + f(k)$$

$$z = \frac{1}{y} f(y + x^2).$$

c. $xz\frac{\partial z}{\partial x} - yz\frac{\partial z}{\partial y} = x^2 - y^2$

Characteristic traces are $\frac{dx}{xz} = \frac{dy}{-yz} = \frac{dz}{x^2-y^2}$

i.e. $\int \frac{1}{x}\,dx = -\int \frac{1}{y}\,dy$

$\ln x = -\ln y + c$

$\Rightarrow xy = c.$

$\cancel{\frac{(x^2-y^2)}{x}dx = \frac{(x^2-y^2)}{-y}dy = z\,dz}$

$$\left.\begin{aligned} \frac{dx}{dt} &= xz \\ \frac{dy}{dt} &= -yz \\ \frac{dz}{dt} &= x^2-y^2 \end{aligned}\right\} \quad x\frac{dx}{dt} + y\frac{dy}{dt} - z\frac{dz}{dt} = 0.$$

i.e. $\frac{1}{2}\frac{d}{dt}\left(x^2+y^2-z^2\right) = 0$

i.e. $x^2+y^2-z^2 = K.$

Then by Lagrange's Method, $x^2+y^2-z^2 = f(xy).$

4. Derivation of D'Alembert's Sol^n:

$$z_{xx} = \frac{1}{c^2} z_{tt}$$

Look for sol^{n}s of form $z = f(x+mt)$

get $f'' = \frac{m^2}{c^2} f''$ $\Rightarrow$ $m = \pm c$

$$z = f(x+ct) + g(x-ct)$$

Make fit initial conditions:

$$F(x) = f(x) + g(x)$$
$$G(x) = cf'(x) - cg'(x)$$

ie. $\int_\alpha^x G(s)\,ds = cf(x) - cg(x)$

$$\Rightarrow 2g(x) = -\frac{1}{c}\int_\alpha^x G(s)\,ds + F(x)$$

$$2f(x) = \frac{1}{c}\int_\alpha^x G(s)\,ds + F(x)$$

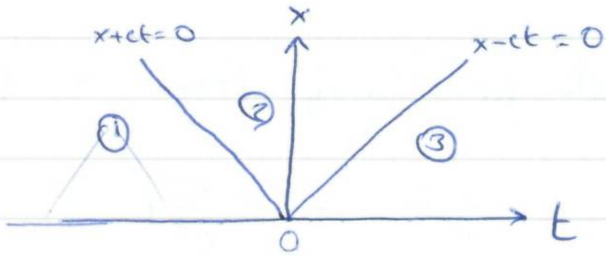
$$\Rightarrow z = \frac{1}{2c}\int_\alpha^{x+ct} G(s)\,ds + \frac{1}{2}F(x+ct)$$

$$- \frac{1}{2c}\int_\alpha^{x-ct} G(s)\,ds + \frac{1}{2}F(x-ct).$$

$\square$.

$F(x) = 0 \qquad G(x) = \begin{cases} -e^{x} & x < 0 \\ e^{-x} & x > 0 \end{cases}$

$z(x,t) = \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} G(s)\,ds$



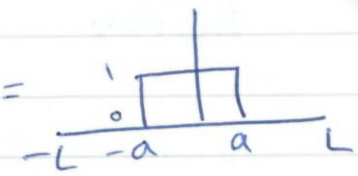
In ① $z(x,t) = \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} -e^{s}\,ds = \dfrac{1}{2c}\left[e^{x-ct} - e^{x+ct}\right]$

② $z(x,t) = \dfrac{1}{2c}\left[\displaystyle\int_{x-ct}^{0} -e^{s}\,ds\right.$

ok.

5. $\theta(x,t)$ $\quad -L \le x \le L \quad t>0.$

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial \theta}{\partial t} \qquad \theta(x,0) =$$



Look for sol(n)s of form $\theta = X(x)T(t)$

$$X''T = \frac{1}{\alpha^2} XT'$$

$$\alpha^2 \frac{X''}{X} = \frac{T'}{T}$$

$$\alpha^2 \frac{X''}{X} = \frac{T'}{T} = \lambda$$

$\lambda>0,\ \lambda=p^2:$ $\quad X'' - \lambda X = 0 \qquad$ ✗ need oscillation.

$\lambda = 0:$ $\quad X = Ax + B \qquad$ b.c.

$\qquad T = C$

$\cancel{T'+pT=0}$

$\cancel{T'=\alpha}$

$$\alpha^2 X'' + p^2 X = 0$$

$\lambda<0\ \lambda=-p^2:$ $\quad X = A\sin\frac{p}{\alpha}x + B\cos\frac{p}{\alpha}x$

$\qquad T = Ce^{-p^2 t}$

b.c. $X'(-L) = X'(L) = 0 \Rightarrow A=0 \qquad B = B = A_0$

b.c. $X'(-L)=X'(L)=0 \Rightarrow A\frac{p}{\alpha}\cos\frac{p}{\alpha}L - B\frac{p}{\alpha}\sin\frac{p}{\alpha}L = 0$

$\qquad A\frac{p}{\alpha}\cos\frac{p}{\alpha}L + B\frac{p}{\alpha}\sin\frac{p}{\alpha}L = 0$

$\Rightarrow B = 0.$

$\hookrightarrow pL = n\pi\alpha$

$\qquad p = \frac{n\pi\alpha}{L}$

$$\Rightarrow \theta = A_0 + \sum_{i=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\exp\left(\frac{-n^2\pi^2\alpha^2 t}{L^2}\right)$$

Before that, at $t=0$.

$$\theta(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x/L)$$

$$\Rightarrow A_0 = \theta(x,0) - \sum_{n=1}^{\infty} A_n \cos(n\pi x/L)$$

Integrate between $-L$ and L. $2LA_0 = \int_{-L}^{L} \theta(x,0) = 2a$

$\Rightarrow A_0 = a/L$

$$\theta(x,0) = \frac{a}{L} + \sum_{n=1}^{\infty} A_n \cos(n\pi x/L)$$

$$\int_{-L}^{L} \theta(x,0) \cos\frac{m\pi x}{L}\, dx = \cancel{\int_{-L}^{L} \frac{a}{L} \cos dx} + A_m L$$

ok.